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MODULE OF HOMEOMORPHISMS TO MODULE

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ABSTRACT
In this paper, after a concise presentation of the modules over rings as a generalization of vector space over the fields, their homeomorphisms are treated. Further builds R-module si R-module of morphisms of the modules.

Keywords:
R-module, left (right) R-module, abelian group, associative ring, R-homeomorphisms

THE MEANING OF THE R-MODULE, FEATURE

Let *M* be an non empty set of equipped with an internal algebraic action [2] marked with the symbol of collection + and *R* an associative ring whatsoever [3]. A set *M* is also equipped with an algebraic external action [2] indicated by the multiplication symbol ·, which, when reflecting $R \times M$ in *M*, is referred to as the left multiplication in *M* with elements from *R*, whereas, when reflecting the $M \times R$ in *M* is called right multiplication in *M* with elements from *R*. In the first case the couple's image $(r, m) \in R \times M$ is written $r{\cdot}m$, in the second case the couple's image $(m, r) \in M \times R$ is written $m{\cdot}r$.

Definition 1.1 [1, 5, 6] In the above conditions, the left module above the *R* ring is called the structure (*M*, +, ·), which has its own attributes:

- (*M*, +) *is an abelian group*; (1)
- $\forall (r_1, r_2, m) \in R^2 \times M$, $r_1(r_2 m) = (r_1 r_2)m$; (2)
- $\forall (r, m_1, m_2) \in R \times M^2$, $r(m_1 + m_2) = rm_1 + rm_2$; (3)
- $\forall (r_1, r_2, m) \in R^2 \times M$, $(r_1 + r_2)m = r_1 m + r_2 m$. (4)

Definition 1.2. Under the above conditions, the right module above the *R* ring is called the structure (*M*, +, ·), which has its own attributes:

- (*M*, +) *is an abelian group*; (1')
- $\forall (m, r_1, r_2) \in M \times R^2$, $(mr_1)r_2 = m(r_1 r_2)$; (2')
- $\forall (m_1, m_2, r) \in M^2 \times R$, $(m_1 + m_2)r = m_1 r + m_2 r$; (3')
- $\forall (m, r_1, r_2) \in M \times R^2$, $m(r_1 + r_2) = mr_1 + mr_2$. (4')

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The left (right) module above the R ring is marked $_R M$ (M_R) and is called R-left module (right). If the left-hand module above R is also the right is called a *module* above the R ring, in short R-module.
If the ring has a single element 1_R (short 1) and the above-mentioned attributes for $_R M$ (M_R) is added the feature

- $\forall m \in M,\ 1 \cdot m = m (m \cdot 1 = m)$ (5)

then the module $_R M$ (M_R) is called *the unitary left (right) module* above the R ring.
In ongoing, the R ring is associated and for a module on such a ring simple naming is used *R-Module*.
Below we will treat the R-modules, implying left R-modules, since the right R-modules are treated analogously.

THEOREM 1.1. A R-module M enjoys the following attributes:

- $\forall m \in M,\ 0_R \cdot m = 0_M$; (6)
- • $\forall r \in R,\ r \cdot 0_M = 0_M$; (7)
- • • $\forall m \in M, \forall r \in R,\ (-r) \cdot m = r \cdot (-m) = -r \cdot m \in M.$ (8)

Proof. Let r be a fixed element of the R ring and m any other element of the $_R M$ module. By Definition 1.1. we have $r \cdot m + 0_R \cdot m = (r + 0_R) \cdot m = r \cdot m$. On the other hand, by the additive group $(M, +)$, we have $r \cdot m + 0_M = r \cdot m$. From here $r \cdot m + 0_R \cdot m = r \cdot m + 0_M$, that gives $0_R \cdot m = 0_M$.

- • $r \cdot 0_M = r \cdot (0_R \cdot m) = (r \cdot 0_R) \cdot m = 0_R \cdot m = 0_M$.
- • • $r \cdot m + (-r) \cdot m = (r + (-r))m = 0_R \cdot m = 0_R \cdot m = 0_M$

$\Rightarrow (-r) \cdot m = -r \cdot m.$

R-HOMEOMORPHISMS OF R-MODULES

Definition 2.1 [1,6] *R-homomorphism (or R-morphism) of a R-module M in a R-module N is called any reflection f: M⟶N having attributes*

- $f(m_1 + m_2) = f(m_1) + f(m_2),\ \forall m_1, m_2 \in M;$ (9)
- $f(r \cdot m) = r \cdot f(m),\ \forall r \in R$ and $\forall m \in M$ (10)

(ose $f(m \cdot r) = f(m) \cdot r,\ \forall r \in R$ and $\forall m \in M$).

If M=N, then the reflection f is called R-endomorphism in M.

THEOREM 2.1. *For every two R-modules M, N, if the reflection f: M⟶N is a R-homomorphism, then*

- $f(0_M) = 0_N$, (11)
- $f(-m) = -f(m), \forall m \in M$, (12)
- $f(m_1 - m_2) = f(m_1) - f(m_2), \forall m_1, m_2 \in M$, (13)

Proof. According to (6) and (10) we have
$f(0_M) = f(0_R \cdot m) = 0_R \cdot f(m) = 0_N$.

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International Journal of Engineering Technology Research & Management

Further, according to (9),

$$0_N = f(0_M) = f(m+(-m)) = f(m) + f(-m),$$

that tells us $f(-m)$ is the symmetric of $f(m)$ in the group $(N, +)$, so $-f(m) = f(-m)$. Finally,

$$f(m_1 - m_2) = f(m_1 + (-m_2)) = f(m_1) + f(-m_2)$$
$$= f(m_1) + (-f(m_2)) = f(m_1) - f(m_2), \forall m_1, m_2 \in M.$$

THEOREM 2.2. *For each two R-modules M, N, reflection p_0: $M \longrightarrow N$, defined by $p_0(m)=0_N$, $\forall m \in M$, is the R-homomorphism of M to N.*

Proof. From the above definition of reflection p_0 we have

$$p_0(m_1+m_2)=0_N = 0_N + 0_N = p_0(m_1) + p_0(m_2), \ \forall m_1, m_2 \in M,$$

which indicates that p_0 enjoys the attribute(9); we also have

$$p_0(r \cdot m)=0_N = r \cdot 0_N = r \cdot p_0(m), \ \forall r \in R \text{ dhe } \forall m \in M,$$

which indicates that p_0 also enjoys the attribute (10).

THEOREM 2.3. Identical reflection $I_M : M \to M$ (e.g the reflection defined by $I_M(m) = m, \forall m \in M$ is an *R*-endomorphism in *M*.

Proof. From the above definition of the identical reflection I_M we have

$$I_M(m_1+m_2)= m_1+m_2 = I_M(m_1) + I_M(m_2), \ \forall m_1, m_2 \in M,$$

Indicating that the I_M enjoys the attribute (9); we also have

$$I_M(r \cdot m)= r \cdot m = r \cdot I_M(m), \ \forall r \in R \text{ dhe } \forall m \in M,$$

which indicates that I_M enjoys the attribute (10).

MODULE $Hom_R(M, N)$ OF *R*-HOMOMORPHISMES OF THE MODULES

The study of homomorphismes of modules bring to the construction of an important module, called the *homomorphism module*.

Let be given the *R*-module *M* and the *R*-module *N*. The set of *R*-homomorphisms from *M* to *N* is written $Hom_R(M, N)$.

Definition 3.1. *Let be f, g two possible reflections from M to N and r an element of an R ring. Then:*

1. Many of the reflection f with the g reflection, which is written f + g, is called reflection f+g: M $\longrightarrow$N, defined by

$$(f+g)(m) = f(m) + g(m) \ , \forall m \in M. \quad (14)$$

2. The opposite reflection of f reflection, which is written -f, is called reflection -f: M $\longrightarrow$N, defined by

$$(-f)(m) = -f(m) \ , \forall m \in M. \quad (15)$$

iJETRM
International Journal of Engineering Technology Research & Management

3. The left product of the reflection f with the element $r \in R$*,, which is written* $r{\cdot}f$ *, is called the reflection* $r{\cdot}f : M \to N$ *, defined by*

$$(r{\cdot}f)(m) = r{\cdot}f(m) \quad , \forall m \in M \; . \qquad (16)$$

An analogy is given to the meaning and the right production $f{\cdot}r$ such that $(f{\cdot}r)(m) = f(m) \cdot r \; , \forall m \in M$.

THEOREM 3.1. *If the reflections f, g are R-homomorphisms from M to N then:*

1. $f{+}g \in Hom_R(M, N)$, (17)

otherwise, their amount f+g is a R-homomorphism from M to N;

2. $-f \in Hom_R(M, N)$, (18)

otherwise, the reverse reflection – f is a R-homomorphism from M to N;

3. *For each r* $\in R$*, where R is commutative,*

$r{\cdot}f \in Hom_R(M, N)$, (19)

otherwise, the left (right) production of f reflection with elements from R is a R-homomorphism from M to N.

Proof.

1. Since the reflections *f, g* are *R*-homomorphisms from *M* to *N*, then

$$(f+g)(m_1+m_2) \overset{(14)}{=} f(m_1+m_2)+g(m_1+m_2)$$

$$\overset{(9)}{=} [f(m_1)+f(m_2)]+[g(m_1)+g(m_2)]$$

$$=[f(m_1)+g(m_1)]+[f(m_2)+g(m_2)]$$

$$\overset{(14)}{=} (f+g)(m_1)+(f+g)(m_2), \; \forall m_1, m_2 \in M,$$

which shows that $f{+}g$ enjoys the attribute (9); we also have

$$(f+g)(rm) \overset{(14)}{=} f(rm)+g(rm)$$

$$\overset{(10)}{=} rf(m)+rg(m)$$

$$= r[f(m)+g(m)]$$

$$\overset{(14)}{=} r[(f+g)(m)], \forall r \in R \text{ dhe } \forall m \in M,$$

which shows that $f{+}g$ enjoys the attribute (10). Consiquently $f{+}g \in Hom_R(M, N)$

2. Reflection *f* is *R*-homomorphism from *M* to *N*, therefore

$$(-f)(m_1+m_2) \overset{(15)}{=} -f(m_1+m_2) \overset{(12)}{=} f(-(m_1+m_2)) = f(-(m_1)+(-m_2))$$

$$\overset{(9)}{=} f(-m_1)+f(-m_2) \overset{(12)}{=} (-f(m_1))+(-f(m_2)) \overset{(15)}{=} (-f)(m_1)+(-f)(m_2), \; \forall m_1, m_2 \in M,$$

which shows that $-f$ enjoys the attribute (9); Also, having in mind and (8) we have

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International Journal of Engineering Technology Research & Management

$$(-f)(r\cdot m) \overset{(12)}{=} f(-r\cdot m) \overset{(8)}{=} f(r\cdot(-m)) \overset{(10)}{=} r\cdot f(-m) \overset{(12)}{=} r\cdot[-f(m)]$$

$$\overset{(15)}{=} r\cdot[(-f)(m)],\ \forall r \in R \text{ dhe } \forall m \in M,$$

which shows that $-f$ enjoys even the attribute (10).

3. We also have

$$(r\cdot f)(m_1+m_2) \overset{(16)}{=} r\cdot f(m_1+m_2) \overset{(9)}{=} r\cdot[f(m_1)+f(m_2)] = r\cdot f(m_1) + r\cdot f(m_2)$$

$$\overset{(16)}{=} (r\cdot f)(m_1) + (r\cdot f)(m_2),\ \forall m_1, m_2 \in M,$$

showing that $r\cdot f$ has its attribute (9); also, knowing that the R ring is commutative we have

$$(r\cdot f)(\rho m) \overset{(16)}{=} r\cdot f(\rho m) \overset{(10)}{=} r\cdot[\rho f(m)] = (r\rho)\cdot f(m) = (\rho r)\cdot f(m)$$

$$= \rho\cdot[r\cdot f(m)] \overset{(16)}{=} \rho\cdot[(r\cdot f)(m)],\ \forall \rho \in R \text{ dhe } \forall m \in M,$$

which shows that $r\cdot f$ also enjoys attribute (10).

Definition 3.2. *R-homomorphism $f+g$: $M \to N$ is called R-homomorphism f: $M \to N$ with R-homomorphism g: $M \to N$, R-homomorphism $-f$ is called the opposite R-homomorphism f: $M \to N$, but R-homomorphism $r\cdot f$ ($f\cdot r$), when R is commutative, is called left (right) production of R-homomorphism f: $M \to N$ with element $r \in R$*

Through this definition, they are introduced into the set $Hom_R(M,N)$ action of addition + and left (right) multiplication, which make it algebra ($Hom_R(M,N)$, +, ·) with two actions.

THEOREM 3.2. If the R ring is commutative, then the algebra ($Hom_R(M,N)$, +, ·) of R-homeomorphisms from M to N is the R-left(right) module.

Proof. We show that they satisfy the conditions (1), (2), (3), (4) of Definition 1.1. of a left R-module.

(1) From the above it is easy to see that:

- $\forall f, g, h \in Hom_R(M,N)$, $(f+g)+h = f+(g+h)$;
- $\forall f \in Hom_R(M,N)$, $f + p_0 = f$;
- $\forall f \in Hom_R(M,N), f+(-f) = p_0$;
- $\forall f, g \in Hom_R(M,N)$, $f+g = g+f$,

indicating that $Hom_R(M,N)$, +) is an abelian group.

(2) $\forall (r_1, r_2, f) \in R^2 \times Hom_R(M,N)$, writing $g = r_2 \cdot f$, we have

$$[r_1\cdot(r_2\cdot f)](m) = (r_1\cdot g)(m) \overset{(16)}{=} r_1\cdot g(m) = r_1\cdot[(r_2\cdot f)(m)] \overset{(16)}{=} r_1\cdot[r_2\cdot f(m)]$$

$$\overset{(10)}{=} r_1\cdot[f(r_2\cdot m)] \overset{(10)}{=} f(r_1\cdot(r_2\cdot m)) = f((r_1\cdot r_2)\cdot m) \overset{(10)}{=} (r_1\cdot r_2)\cdot f(m)$$

$$\overset{(16)}{=} [(r_1\cdot r_2)\cdot f](m),\ \forall m \in M,$$

which indicates that $r_1\cdot(r_2\cdot f) = (r_1\cdot r_2)\cdot f$.

(3) $\forall (r, f, g) \in R \times [Hom_R(M,N)]^2$ we have

iJETRM
International Journal of Engineering Technology Research & Management

$$[r\cdot(f+g)](m) \overset{(16)}{=} r\cdot[(f+g)(m)] \overset{(14)}{=} r\cdot[f(m)+g(m)]=r\cdot f(m)+r\cdot g(m)$$
$$\overset{(16)}{=} (r\cdot f)(m)+(r\cdot g)(m) \overset{(14)}{=} (r\cdot f+r\cdot g)(m),\ \forall m \in M,$$

which indicates that $r\cdot(f+g)= r\cdot f+r\cdot g$.

(4) $\forall(r_1,r_2, f) \in R^2 \times Hom_R(M,N)$, we have

$$[(r_1+r_2)\cdot f](m) \overset{(16)}{=} (r_1+r_2)\cdot f(m) \overset{(10)}{=} f((r_1+r_2)m)=f(r_1m+r_2m) \overset{(9)}{=} f(r_1m)+f(r_2m)$$
$$\overset{(10)}{=} r_1\cdot f(m)+r_2\cdot f(m) \overset{(16)}{=} (r_1\cdot f)(m)+(r_2\cdot f)(m) \overset{(9)}{=} (r_1\cdot f+r_2\cdot f)(m),\ \forall m \in M,$$

which indicates that $(r_1+r_2)\cdot f = r_1\cdot f+r_2\cdot f$.

Analogously it is shown that ($Hom_R(M,N)$, +, ·) is the right *R*-module when · is right multiplication with elements from *R*.

REFERENCES

[1] Kristaq Filipi, Leksione të shkruara për modulet, Tiranë 2015.
[2] Kristaq Filipi, Algjebra dhe Gjeometria(ribotim), Tiranë 2015.
[3] Kristaq Filipi, Algjebër Abstrakte, Tiranë 2013.
[4] M.Hazewinkel etj., Algebras, Rings and Modules, 2004.
[5] E.Ademaj, E. Gashi, Algjebra e Përgjithshme, Prishtinë 1986.
[6] B. Gazidede, Algjebra 1, Tiranë 2006.
[7] Y.Bahturin, Osnovnie strukturi sovremenoj allgebri, Moske 1990.
[8] Remy Oudompheng, Duality and canonical modules,Spring school in Local Algebra 2009.
[9] Eisenbud D., Commutative Algebra with a view Toward Algebraic Geometry, Springer-Verlag New York, 1995.
[10] Saunders, MacLane, Homology, Springer –Verlag Berlin 1963.